Math 245C Lecture 16 Notes

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1 Smooth Density Results, Smooth Urysohn's Lemma, and Characters of $\mathbb{R}^n$

1.1 Density results for C_c^∞ and $\mathcal{S}$

Let $\phi_1 \in C_c^\infty(\overline{B_1(0)})$ be such that $\phi_1 > 0$ on $B_1(0)$ and such that

$$\int_{\mathbb{R}^n} \phi_1(x)\, dx = 1.$$

For example, take

$$\phi(x) := \begin{cases} e^{1/(\|x\|^2-1)} & |x| < 1 \\ 0 & |x| \geq 1 \end{cases}, \qquad \phi_1(x) = \frac{\phi(x)}{\int \phi}.$$

Lemma 1.1. *If $1 \leq p < \infty$, then C_c^∞ and $\mathcal{S}$ are dense in L^p.*

Proof. Let $f \in L^p$, and let $\varepsilon_0 > 0$. We are to find $g \in C_c^\infty$ such tht $\|f - g\|_p < \varepsilon_0$. Choose $\tilde{g} \in C_c$ such that $\|\tilde{g} - f\|_p < \varepsilon_0/2$. Set

$$\phi_\varepsilon(x) = \frac{1}{\varepsilon^n}\phi_1\left(\frac{x}{\varepsilon}\right).$$

We have $\phi_\varepsilon * \tilde{g} \in C^\infty$. Furthermore, $\operatorname{supp}(\phi_\varepsilon * \tilde{g}) \subseteq \operatorname{supp}(\tilde{g}) + \overline{B_\varepsilon(0)}$. Hence, $\phi_\varepsilon * \tilde{g} \in C_c^\infty$. Choose ε small enough such that

$$\|\phi_\varepsilon * \tilde{g} - \tilde{g}\|_p < \varepsilon/2.$$

So the desired inequality holds for $g = \phi_\varepsilon * \tilde{g}$. In conclusion, $L^p \subseteq \overline{C_c^\infty}^{L^p} \subseteq \overline{\mathcal{S}}^{L^p} \subseteq L^p$. $\square$

Lemma 1.2. *C_c^∞ and $\mathcal{S}$ are dense in $C_0(\mathbb{R}^n)$ for the uniform norm.*

Proof. Let $f \in C_0$. Recall that $\overline{C_c}^{L^\infty} = C_0$. Hence, given $\varepsilon_0 > 0$, there is a $\tilde{g}\ inC_c$ such that $\|f - \tilde{g}\|_u \leq \varepsilon/2$. Since $\tilde{g}$ is uniformly continuous and bounded,

$$\lim_{\varepsilon \to 0} \|\tilde{g} - \phi_\varepsilon * \tilde{g}\|_u = 0.$$

Thus , there is $\varepsilon > 0$ such that

$$\|\tilde{g} - \phi_\varepsilon * \tilde{g}\|_u \leq \varepsilon/2.$$

So we get

$$\|f - g\|_u \leq \varepsilon.$$

If we set $g = \phi_\varepsilon * \tilde{g}$, as before, $g \in C_c^\infty$. In conclusion, $C_0 \subseteq \overline{C_c^\infty}^{L^\infty} \subseteq \overline{\mathcal{S}}^{L^\infty} \subseteq C_0$. □

1.2 Smooth Urysohn's lemma

Lemma 1.3 (Urysohn)**.** *Let $K \subseteq \mathbb{R}^n$ be a compact nonempty set, and let $U \subseteq \mathbb{R}^n$ be an open set such that $K \subseteq U$. Then there exists a function $f \in C_c^\infty$ such that $f|_K = 1$,* $\mathrm{supp}(f) \subseteq U$*, and* $\mathrm{supp}(f)$ *is compact.*

This is useful on manifolds. Treat a neighborhood of a point as a subset of $\mathbb{R}^n$. If you want to integrate a function on the manifold, you can integrate it over every neighborhood.

Proof. Set $3\delta = \mathrm{dist}(K, U^c) > 0$. Let $K_1 = \{x \in \mathbb{R}^n : \mathrm{dist}(x, K) \leq \delta\}$. Note that K_1 is compact, and $\mathrm{dist}(K_1, U^c) \geq \delta$. Let $f = \phi_\varepsilon * \mathbb{1}K_1$. Then $f \in C^\infty$, and $\mathrm{supp}(f) \subseteq K_1 + \overline{B_\varepsilon(0)} \subseteq U$ if $\varepsilon < \delta$. So $f \in C_c^\infty$, and f has compact support. If $x \in K$, then

$$f(x) = \int_{\mathbb{R}^n} \mathbb{1}_{K_1}(x - y)\phi_\varepsilon(y)\, dy = \int_{\overline{B_\varepsilon(0)}} \mathbb{1}_{K_1}(x - y)\phi_\varepsilon(y)\, dy.$$

If $x \in K$ and $|y| \leq \varepsilon$, then $x - y \in K_1$. So

$$f(x) = \int_{\overline{B_\varepsilon(0)}} \phi_\varepsilon(y)\, dy = \int_{\mathbb{R}^n} \phi_\varepsilon(y)\, dy = 1.$$ □

1.3 Characters on $(\mathbb{R}^n, +)$

Proposition 1.1. *Let $\phi : \mathbb{R}^n \to \mathbb{C}$ be a measurable function such that $|\phi(x)| = 1$ and $\phi(x + y) = \phi(x)\phi(y)$ for any $x, y \in \mathbb{R}^n$. Then*

1. *There exists $\xi \in \mathbb{R}^n$ such that $\phi(x) = e^{2\pi i \xi \cdot x}$.*

2. *If we further assume that ϕ is $\mathbb{Z}^n$-periodic, then $\xi \in \mathbb{Z}^n$.*

Proof. Let $(e_j)_{j=1}^n$ the standard orthonormal basis of $\mathbb{R}^n$. If $x \in \mathbb{R}^n$, then

$$\phi(x) = \phi\left(\sum_{j=1}^n x_j e_j\right) = \prod_{i=1}^n \phi(x_j e_j).$$

The function $x \mapsto \phi(x, e_j)$ can be identified with the restriction of ϕ to a 1-dimensional space. Therefore, $t \mapsto \phi(te_j)$ satisfies the assumption of the lemma for $n = 1$. It is not a loss of generality to assume $n = 1$.

Set

$$F(x) = \int_0^x \phi(t)\, dt.$$

There exists $a \neq 0$ such that $F(a) \neq 0$ (lest $\phi \equiv 0$). Set $A = 1/F(a)$. We have

$$\begin{aligned}\phi(a) &= \phi(x)A\int_0^a \phi(t)\, dt = A\int_0^a \phi(x)\phi(t)\, dt = A\int_0^a \phi(x+t)\, dt = A\int_x^{x+a} \phi(z)\, dz \\ &= A(F(x+a) - F(x)).\end{aligned}$$

Since F is continuous, this gives us that ϕ is continuous. Since ϕ is continuous, F is continuously differentiable. Apply the equality above again to conclude that $\phi \in C^1$. Differentiating both sides, we obtain

$$\phi'(x) = A(\phi(x+a) - \phi(x)) = A(\phi(x)\phi(a) - \phi(x)) = A\phi(x)(\phi(a) - 1) = \phi(x)B.$$

So we get

$$\frac{d}{dx}\ln(\phi(x)) = \frac{\phi'(x)}{\phi(x)} = B.$$

Integrating, we obtain

$$\phi(x) = \phi(0)e^{Bx}.$$

But $\phi(0) = \phi(0+0) = \phi(0)^2$, and so $\phi(0) \in \{0, 1\}$. Since $|\phi(0)| = 1$, we conclude that $\phi(0) = 1$. Write $B = B_1 + iB_2$ with$B_1, B_2 \in \mathbb{R}$. Then

$$\phi(x) = e^{B_1 x}e^{iB_2 x},$$

and

$$1 = |\phi(1)| = e^{B_1} \implies B_1 = 0.$$

So $\phi(x) = e^{2\pi i \xi x}$, where$\xi = B_2/(2\pi)$.

Assume ϕ is $\mathbb{Z}$-periodic. Then

$$e^{2\pi i \xi} = \phi(1) = \phi(0) = 1.$$

So $2\pi i \xi \in 2\pi\mathbb{Z}$. That is, $\xi \in 2\pi\mathbb{Z}$. □